Math 'Convincing and Proving' Critiquing 'Proofs' Tasks - Set #4

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The aim of this assessment is to provide the opportunity for you to:
- evaluate 'proofs' of given statements and identify which are correct;
- identify errors in 'proofs'.

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1. The Pythagorean' Theorem

Here are three attempts to prove the Pythagorean theorem.

When you add add three consecutive numbers, your answer is always a multiple of three.

Look carefully at each attempt. Which is the best 'proof'? Explain your reasoning as fully as possible.
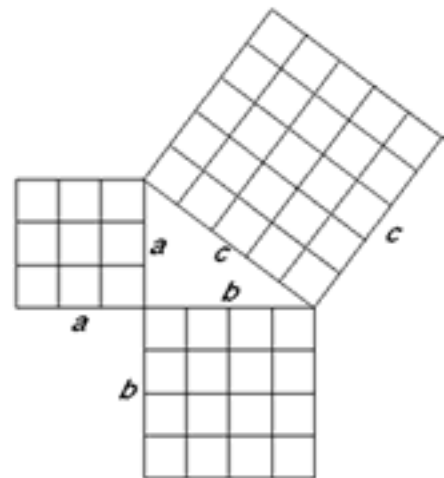
Attempt 1:

Suppose a right-angled triangle has sides of length a, b and c

Draw squares on the three sides as shown.

Divide these squares into smaller squares.

You can see that the number of squares on the two shorter sides add up to make the number of squares on the longest side.

So: $a^2 + b^2 = c^2$

Attempt 2:

Suppose that you start with **four** right angled triangles with sides of length a, b and c and a square tray with sides of length a + b.

You can arrange the triangles into the tray in two different ways as shown here.

In the first way, you leave two square holes. These have a combined area of $a^2 + b^2$.

In the second way you leave one large square hole. This has an area of c^2.

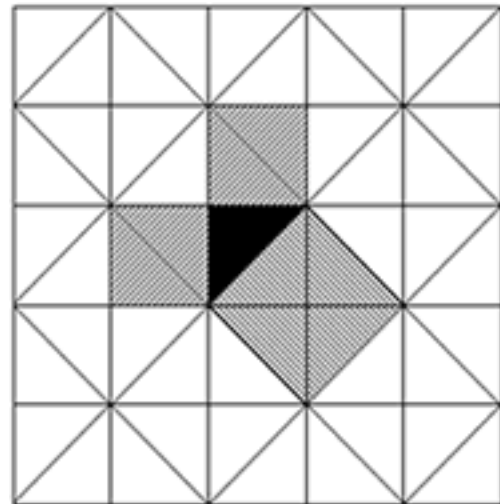Since these areas are equal
$a^2 + b^2 = c^2$
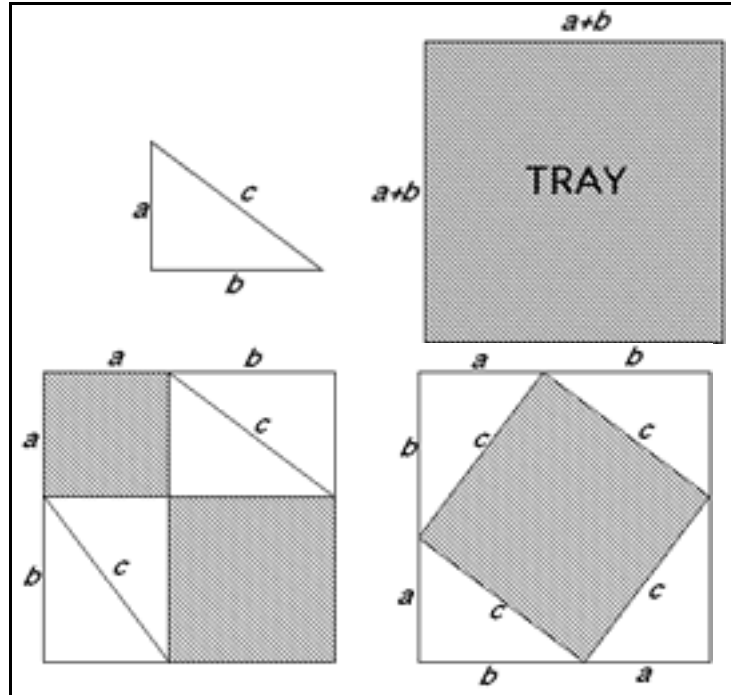
Attempt 3:

The proof of the Pythagorean theorem is clear from this diagram.

The squares on the two shorter sides of the black triangle are each made from two congruent triangles.

These fit together to make the square on the longest side - the hypotenuse.

The best proof is attempt number ....

This is because ....

My criticisms of the others are...........

2. Means

Here are two attempts to prove the following result about the arithmetic mean and geometric mean of two numbers:

For any positive numbers a and b,
$$\frac{a+b}{2} \geq \sqrt{ab}$$

Look carefully at each attempt. Which is the best 'proof'? Explain your reasoning as fully as possible.

Attempt 1:

Assuming that
$$\frac{a+b}{2} \geq \sqrt{ab}$$

Multiply both sides by 2
$$a+b \geq 2\sqrt{ab}$$

Squaring
$$(a+b)^2 \geq 4ab$$

$$a^2 + b^2 + 2ab \geq 4ab$$

$$a^2 + b^2 - 2ab \geq 0$$

$$(a-b)^2 \geq 0$$

Which is true for positive numbers.

So the assumption was true.

Attempt 2:

For all positive numbers
$$(\sqrt{a} - \sqrt{b})^2 \geq 0$$

$$a - 2\sqrt{a}\sqrt{b} + b \geq 0$$

$$a + b \geq 2\sqrt{a}\sqrt{b}$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

So the result is true.

Attempt 3:

Look at the diagram.

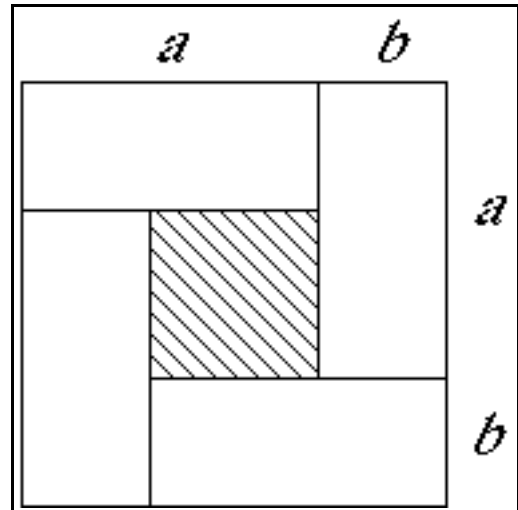The area of the large square $= (a+b)^2$

The unshaded area $= 4ab$ (2)

Since (1) is greater than (2)

$(a+b)^2 > 4ab$ or

$(a+b) > 2\sqrt{ab}$ or

$\dfrac{a+b}{2} > \sqrt{ab}$

So the result is true

The best proof is attempt number ....

This is because ....

My criticisms of the others are...........

3. Even and Odd

Here are three attempts to prove the following result for natural numbers.

If n^2 is an odd number, then n is an odd number.

Look carefully at each attempt. Which is the best 'proof'? Explain your reasoning as fully as possible.

Attempt 1:
If n was an even number, then n=2m for some m.

Then $n^2 = 4m^2 = 2(2m^2)$ will also be an even number.

So as n^2 is an odd number, then n must be an odd number.

Attempt 2:

If n was an odd number, then since odd x odd = odd, n^2 is also odd.

Then $n^2 + n$ would be the sum of two odd integers and would therefore be even.

Since $n^2 + n = n(n + 1)$ it is also the product of two consecutive numbers and so it *certainly is* even. Therefore n is odd.

Attempt 3:

Since n^2 is odd, we have $n^2 = 4m^2 + 4m + 1$ for some m.

This means $n^2 = (2m + 1)^2$ for some m

so n = 2m + 1 which is odd.

The best proof is attempt number ....

This is because ....

My criticisms of the others are...........