

Math 'Convincing and Proving' Critiquing 'Proofs' Tasks - Set #4 (solutions)

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The aim of this assessment is to provide the opportunity for you to:

- evaluate 'proofs' of given statements and identify which are correct;
- identify errors in 'proofs'.

1. The Pythagorean' Theorem

Here are three attempts to prove the Pythagorean theorem.

When you add add three consecutive numbers, your answer is always a multiple of three.

Look carefully at each attempt. Which is the best 'proof'? Explain your reasoning as fully as possible.

Attempt 1:

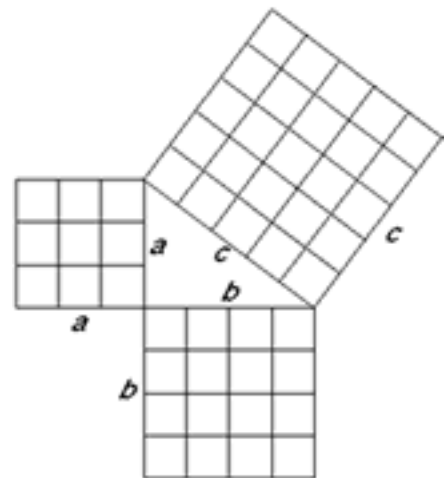
Suppose a right-angled triangle has sides of length a , b and c

Draw squares on the three sides as shown.

Divide these squares into smaller squares.

You can see that the number of squares on the two shorter sides add up to make the number of squares on the longest side.

So: $a^2 + b^2 = c^2$



Attempt 2:

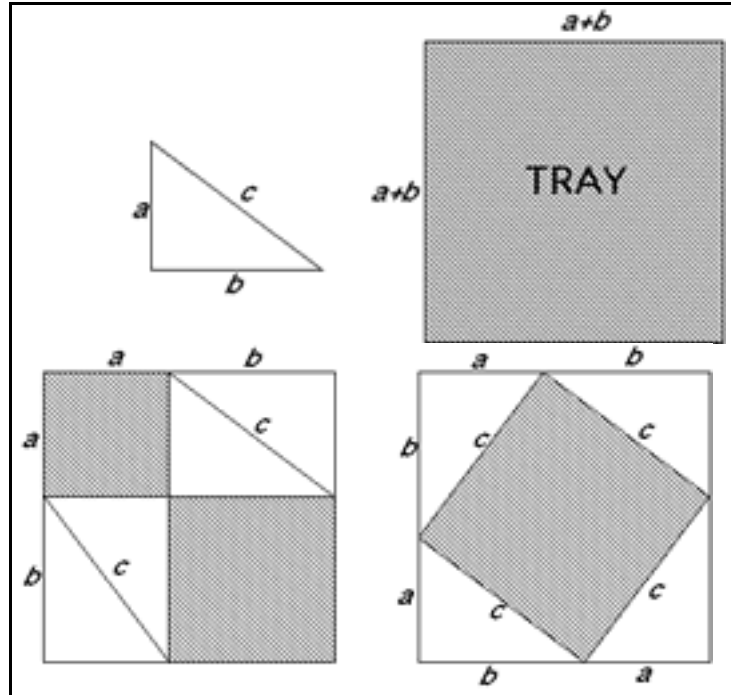
Suppose that you start with **four** right angled triangles with sides of length a , b and c and a square tray with sides of length $a + b$.

You can arrange the triangles into the tray in two different ways as shown here.

In the first way, you leave two square holes. These have a combined area of $a^2 + b^2$.

In the second way you leave one large square hole. This has an area of c^2 .

Since these areas are equal
 $a^2 + b^2 = c^2$

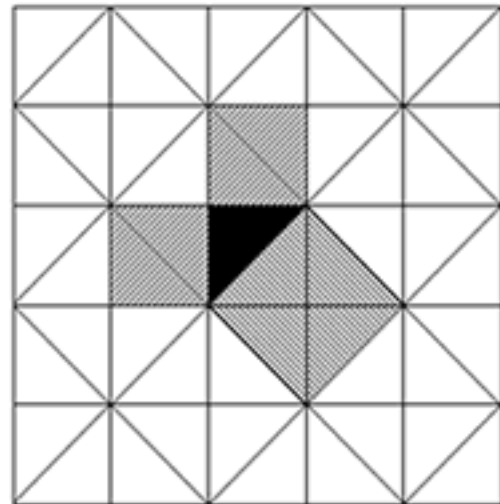


Attempt 3:

The proof of the Pythagorean theorem is clear from this diagram.

The squares on the two shorter sides of the black triangle are each made from two congruent triangles.

These fit together to make the square on the longest side - the hypotenuse.



The best proof is attempt number

This is because

My criticisms of the others are.....

Sample Solution:

Attempts 1 and 3 provide proofs for special cases only.

Attempt 1 is a sufficient demonstration where the sides of the triangle are in the ratio 3:4:5

Attempt 3 is sufficient when the sides are in the ratio 1:1: $\sqrt{2}$

Attempt 2 provides the only dissection which works for any right angled triangle.

2. Means

Here are two attempts to prove the following result about the arithmetic mean and geometric mean of two numbers:

For any positive numbers a and b,

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Look carefully at each attempt. Which is the best 'proof'? Explain your reasoning as fully as possible.

Attempt 1:

Assuming that

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Multiply both sides by 2

$$a+b \geq 2\sqrt{ab}$$

Squaring

$$(a+b)^2 \geq 4ab$$

$$a^2 + b^2 + 2ab \geq 4ab$$

$$a^2 + b^2 - 2ab \geq 0$$

$$(a-b)^2 \geq 0$$

Which is true for positive numbers.

So the assumption was true.

Attempt 2:

For all positive numbers

$$(\sqrt{a} - \sqrt{b})^2 \geq 0$$

$$a - 2\sqrt{a}\sqrt{b} + b \geq 0$$

$$a + b \geq 2\sqrt{a}\sqrt{b}$$

$$\frac{a + b}{2} \geq \sqrt{ab}$$

So the result is true.

Attempt 3:

Look at the diagram.

The area of the large square = $(a + b)^2$

The unshaded area = $4ab$ (2)

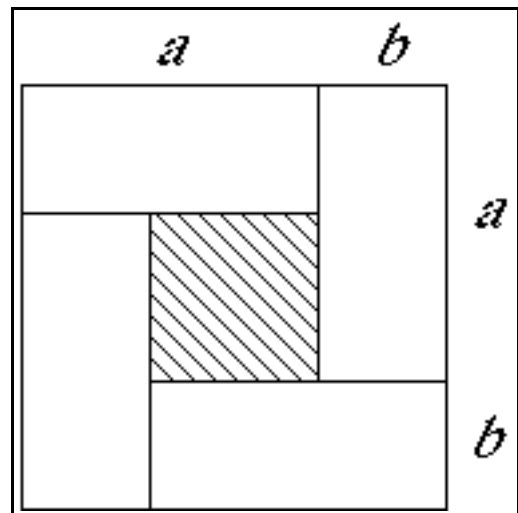
Since (1) is greater than (2)

$$(a + b)^2 > 4ab \text{ or}$$

$$(a + b) > 2\sqrt{ab} \text{ or}$$

$$\frac{a + b}{2} > \sqrt{ab}$$

So the result is true



The best proof is attempt number

This is because

My criticisms of the others are.....

Sample Solution:

Attempt 1, although a logical approach, assumes what we are setting out to prove. If the argument were reversed, it would provide the basis for a proof.

Attempt 2 is correct.

Attempt 3 is the basis for a very elegant proof, but there are some holes and unnecessary jumps in it at present. The first is that it is always dangerous to argue from a diagram because the diagram does not show all possible cases. In this example, the shaded area is clearly greater than 0, but what if a and b were equal? In this case, the large square would be made entirely of four small squares and the shaded area would be zero. The diagram currently assumes that $a > b$. The special case when $a = b$ needs to be considered. The final result thus should become $\frac{a+b}{2} \geq \sqrt{ab}$ instead of $\frac{(a+b)}{2} > \sqrt{ab}$. The diagram suggests that the general identity $(a+b)^2 = (a-b)^2 + 4ab$ would explain why $(a+b)^2 \geq 4ab$ for all possible values of a and b .

3. Even and Odd

Here are three attempts to prove the following result for natural numbers.

If n^2 is an odd number, then n is an odd number.

Look carefully at each attempt. Which is the best 'proof'? Explain your reasoning as fully as possible.

Attempt 1:

If n was an even number, then $n=2m$ for some m .

Then $n^2 = 4m^2 = 2(2m^2)$ will also be an even number.

So as n^2 is an odd number, then n must be an odd number.

Attempt 2:

If n was an odd number, then since odd \times odd = odd, n^2 is also odd.

Then $n^2 + n$ would be the sum of two odd integers and would therefore be even.

Since $n^2 + n = n(n+1)$ it is also the product of two consecutive numbers and so it *certainly is* even. Therefore n is odd.

Attempt 3:

Since n^2 is odd, we have $n^2 = 4m^2 + 4m + 1$ for some m .

This means $n^2 = (2m+1)^2$ for some m

so $n = 2m+1$ which is odd.

The best proof is attempt number

This is because

My criticisms of the others are.....

Sample Solution:

Attempt 1 is correct.

Attempt 2 assumes what we are asked to prove (see the first line). It could, however, be modified to form a proof in the following way:

Take any positive integer n .

Let $y = n(n + 1) = n^2 + n$.

n or $n+1$ must be even since they are consecutive numbers.

So y must be the product of an odd and even number. It is therefore even.

So $n^2 + n$ must be even.

Now if n^2 is odd, then $n (= y - n^2)$ must also be odd. (Since even-odd = odd)

Attempt 3 does not justify why *every* odd square number can be expressed as $4m^2 + 4m + 1$ for some m .